PMM U.S.S.R.,Vol.49,No.4,pp.434-440,1985
Printed in Great Britain 0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

KINETIC THEORY AND THE PRINCIPLE OF MATERIAL FRAME-INDIFFERENCE IN THE MECHANICS OF CONTINUOUS MEDIA*

V.S. GALKIN and V.I. NOSIK

The property of Boltzmann's kinetic equation in the presence of Euclidean transformations such as non-stationary rotations and translations of the frame of reference is discussed. It is shown that in the transition from an inertial frame to a rotating frame, in Boltzmann's equation additional intertial terms appear, and in the transition from non-inertial to non-inertial, the equation is invariant under the above transitions. The algorithms, and the results of an approximate method for solving the equation, of the Chapman-Enskog method in particular, are also invariant. The additional terms appear, specifically, in the expressions for stresses and the heat fluxes in Barnett's approximation, and in this sense these expressions are frame-dependent. Because of the condition that Knudsen's numbers should be small, this limits the domain of applicability of one of the basic postulates of the axiomatic theory of continuous media, namely the principle of material frame-indifference (or the principle of material objectivity), in accordance with which the constitutive (determining) relations should be invariant under continuous changes of the frame of reference. The existing papers on this subject are critically analysed.

For more than ten years a discussion /1—7/ has been going on as a consequence of the well-known fact that the formulae for the stresses P_{ij} and heat fluxes q_i in Barnett's approximation contain, as cofactors, the components of the spin tensor

$$\Omega_{km} = {}^1/_2\,(\partial u_k/\partial x_m - \partial u_m/\partial x_k)$$

and consequently these formulae are invariant under a Euclidean type of transformation (see /8/). In other words, in the expressions for P_{ij} and q_i, when passing from an inertial to a non-inertial frame, additional terms appear, i.e. the consequence of the frame being non-inertial will not be merely the appearance of 'Eulerian' inertia forces in the equation of momentum.

Hence, it was initially concluded in /1—3/ that the principle of material frame-indifference was limited (see 9, 10/). The invariant terms in P_{ij} and q_i were indicated in /2/. The properties of the Maxwell transfer equation for P_{ij} and q_i, and the iteration method for their solution in a non-inertial frame of reference were studied; it was remarked that the non-invariance in the sense mentioned above is due to the action of microscopic Coriolis forces. The action of the latter was studied in more detail in /3/. Some inaccuracies allowed in /1, 2/ were corrected in /5/ (see Sect.6 below).

Subsequently, the question of applying the conclusions which follow from kinetic theory to the principle of material frame-indifference /4—6/, and the properties of Boltzmann's equation in the presence of the Euclidean transformations were studied /4/. An incorrect deduction that the solutions of Boltzmann's equations satisfy the principle was made: this even led to the conclusion that the higher approximations, starting with Barnett's, of the Chapman-Enskog and Maxwell methods for solving Boltzmann's equation are incorrect as the Knudsen number $\mathrm{Kn} \to 0$ /6/.

Barnett's approximation for P_{ij} and q_i was again considered in /7/ and it was suggested that the principle of material frame-indifference should be generalized (see Sect.7 below).

The foundations of the mechanics of continuous media, and their connection with the kinetic theory are of great importance, /11/. Papers /1—7/ are concerned precisely with discussions on these subjects. However, a sufficiently full treatment of the qeustions was not given, and some conclusions were false to varying degrees (especially in /6/). This was due to confusion in the definitions.

The aim of the present paper is to analyse successively the questions formulated in /1—7/. The properties of Boltzmann's equation in the presence of Euclidean transformations, then of the algorithm of the Chapman-Enskog method, and the stresses in the heat flux in the Barnett approximation are studied. As a result, it is concluded that there are no contradictions between the properties of the equation and its approximate solutions. Some examples and the domain of applicability of the principle of material frame-indifference are discussed.

*Prikl.Matem.Mekhan.,49,4,563-571,1985

1. Let us introduce some definitions and notation /1, 10/. A vector, its components and the column-matrix of the latter are denoted by the same lower-case letters (for example, $\mathbf{x}, x_i, x$), and the similar quantities for the second-rank tensor are denoted by the same upper-case letters (for example, $\mathbf{P}, P_{ij}, P$), with $i, j = 1, 2, 3$. Summation is carried out with respect to the repeating indices, and the matrix formulae for transforming vectors and tensors are used when the frame of reference is changed.

Let $\Sigma, \Sigma^*, \Sigma^\circ$ be orthonormalized Cartesian frames of reference, only Σ being inertial. The coordinates in these frames are connected by the transformations

$$x^* = R^*(t)x + b^*(t), \quad x^\circ = R^\circ(t)x + b^\circ(t) \tag{1.1}$$

$$t^* = t + d^*, \quad t^\circ = t + d^\circ$$

The matrices R^* and R° which describe the rotation are orthogonal ($RR^T = R^TR = I$, where I is the unit matrix). For brevity, we set $b_i(t) = 0, d = 0$, since the frame-dependence of P_{ij} and q_i is due to its unsteady rotation /1–7/, and the conclusions reached below do not depend on this assumption.

The coordinates of the Σ°- and Σ^*-systems are connected by the relations

$$x^\circ = Qx^*, \quad Q = R^\circ R^{*T} \tag{1.2}$$

The matrices of the angular velocities of the coordinate axes of these systems relatively to Σ are

$$W^* = R^{*\cdot}R^{*T}, \quad W^{\circ} = R^{\circ\cdot}R^{\circ T}, \quad W^\circ = QW^*Q^T + Q^\cdot Q^T \tag{1.3}$$

(the dot means differentiation with respect to time t).

By (1.1), for the velocities of the molecules $\xi = \mathbf{x}^\cdot, \xi^* = \mathbf{x}^{*\cdot}, \xi^\circ = \mathbf{x}^{\circ\cdot}$ we have

$$\xi^* = R^*\xi + R^{*\cdot}x, \quad \xi^\circ = Q\xi^* + Q^\cdot x^* \tag{1.4}$$

From (1.1)–(1.4) we have the following expressions for the accelerations of the molecules:

$$\xi^{*\cdot} = R^*F(x^*, t) + 2W^*\xi^* + W^{*\cdot}x^* - W^*W^*x^* \tag{1.5}$$

$$\xi^{\circ\cdot} = R^\circ F(x^\circ, t) + 2W^\circ\xi^\circ + W^{\circ\cdot}x^\circ - W^\circ W^\circ x^\circ$$

Here $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$ is the external force referred to the mass of a molecule m, and the remaining terms correspond to the Coriolis rotational and centripetal accelerations.

Following /1, 6, 7/, we describe the n-rank tensor with components $a_{i_1 i_2 \ldots i_n}$ as frame-indifferent if for (1.1) we have $a^*_{i_1 \ldots i_n} = R^*_{i_1 j_1} \ldots R^*_{i_n j_n} a_{j_1 \ldots j_n}$, for example, for a frame-indifferent second-rank tensor $A^* = R^*AR^{*T}$ for a vector, $u^* = R^*u$, or for a scalar $a^*(\mathbf{x}^*, t) = a(\mathbf{x}, t)$.

As shown in /4, 6/, the velocity distribution function of the molecules $f = f(\xi, \mathbf{x}, t)$, the element of velocity space $d\xi$ and, therefore, the gas density ρ are frame-indifferent scalars. Since

$$\mathbf{u} = \frac{m}{\rho}\int \xi f \, d\xi$$

by virtue of (1.4) the macroscopic velocity $\mathbf{u}$ is not a frame-indifferent vector, because

$$u^* = R^*u + R^{*\cdot}x, \quad u^\circ = Qu^* + Q^\cdot x^* \tag{1.6}$$

The tensor $\mathbf{V}$ with components $V_{ij} = \partial u_i/\partial x_j$, also is not frame-indifferent since, by (1.6), (1.3), we have

$$V = R^{*T}(V^* - W^*)R^* = R^{\circ T}(V^\circ - W^\circ)R^\circ \tag{1.7}$$

At the same time, a tensor with components which are derivatives of V_{ij} with respect to x_k is frame-indifferent.

We emphasize that for the tensor $\mathbf{V}^* - \mathbf{W}^*$, from (1.7) we have

$$V^* - W^* = Q^T(V^\circ - W^\circ)Q \tag{1.8}$$

By (1.4) and (1.6), the vector of the molecule's own velocity $\mathbf{c} = \xi - \mathbf{u}$ ($c^* = R^*c$, $c^\circ = Qc^*$) is frame-indifferent. Therefore, the central moments of the distribution function

$$M^{(n)} = m\int \mathbf{c}^{(n)} f \, d\xi \equiv [\mathbf{c}^{(n)}] \tag{1.9}$$

and the derivatives with respect to their coordinates, in particular the stress tensor $\mathbf{P}$ with the components $P_{ij} = [c_ic_j - {}^1/_3\delta_{ij}c^2]$, the pressure $p = {}^1/_3[c^2]$, and the heat flux $\mathbf{q} = {}^1/_2[\mathbf{c}c^2]$, are frame-indifferent as well.

2. The fact that the frame-indifference of $\mathbf{P}$ and $\mathbf{q}$ follows from the kinetic theory led Speziale /6/ to a false conclusion: the presence of the terms with factors Ω_{km} in Barnett's stresses $P_{ij}^{(2)}$ and the heat fluxes $q_i^{(2)}$ are allegedly a consequence of the inaccuracy of the

approximate methods in kinetic theory, in particular, of the Chapman-Enskog method. However, generally speaking, it does not follow from the frame-indifference of the tensors that the formulae which express these tensors by others are invariant under the transformations (1.1) (the scaling rules for the frame-indifferent quantities, and not the dependence of these quantities on the space derivatives of macroparameters are frame-independent). The requirement of such an invariance is an additional postulate of the material frame-indifference principle /9, 10/. Below we show that the violation of this principle in $P_{ij}^{(2)}, q_i^{(2)}$ is a consequence of the corresponding properties of Boltzmann's equation.

The false conclusion made in /6/ is connected also with the faulty interpretation of the findings in /4/.

3. The findings of /4/ consist of the following. Boltzmann's equation in an inertial Σ-system has the form

$$\frac{\partial f}{\partial t} - \xi_j \frac{\partial f}{\partial x_j} + \xi_j^{\cdot} \frac{\partial f}{\partial \xi_j} = J(f), \quad \xi_j^{\cdot} = F_j(\mathbf{x}, t) \tag{3.1}$$

The quantity $\mathbf{F}$ was assumed /4/ to be an arbitrary function of $\boldsymbol{\xi}$. However, then the third term on the left in Eq. (3.1) should be expressed /12/ in the form $\partial(\xi_j^{\cdot} f)\partial\xi_j$.

The collision integral $J(f)$ is invariant under the Euclidean group of transformations, /4/. In fact, it equals the difference between the "increase" and "decrease" of the number of molecules in the element of phase volume which is invariant under the transformations (1.1), the difference being caused by the instantaneous "point" collisions of molecules. This property is particularly evident for models of the collision integral of the relaxation type.

In a non-inertial Σ^*-system, Eq. (3.1), taking (1.1) and (1.4) into account, can be reduced to the form

$$\frac{\partial f^*}{\partial t} - \xi_j^* \frac{\partial f^*}{\partial x_j^*} - \xi_j^{*\cdot} \frac{\partial f^*}{\partial \xi_j^*} = J^*(f^*) \tag{3.2}$$

The quantity $\xi_j^{*\cdot}$ is given by (1.5). In a Σ°-system we obtain the same equation by replacing the asterisks by degrees. Thus, Boltzmann's equation maintains its form, and in this sense it is invariant under the Euclidean transformations /4/. Clearly, in the same sense the equation of momentum, which is obtained by multiplying Eqs. (3.2) by $m\xi_i^*$ and integrating over the whole velocity space,

$$a_i^* = F_i^* - \frac{1}{\rho^*} \frac{\partial T_{ik}^*}{\partial x_k^*} \quad \left(a_i^* \equiv \frac{D^* u_i^*}{Dt}\right) \tag{3.3}$$

$$F_i^* = F_i(\mathbf{x}^*, t) - 2W_{ij}^* u_j^* - W_{ik}^* W_{km}^* x_m^* - W_{ik}^{*\cdot} x_k$$

$$\frac{D^*}{Dt} = \frac{\partial}{\partial t} - u_j^* \frac{\partial}{\partial x_j^*}, \quad T_{ij} = P_{ij} - p\delta_{ij}, \quad p = \rho \frac{k}{m} T$$

is also invariant.

On multiplying (3.2) by m and $\frac{1}{2}mc^{*2}$, and integrating over the velocities, we obtain the continuity and energy equations

$$\frac{D^*\rho^*}{Dt} - \rho^* V_{jj}^* = 0, \quad \frac{3}{2} n^* k \frac{D^* T^*}{Dt} - T_{ij}^* V_{ij}^* - \frac{\partial q_j^*}{\partial x_j^*} = 0 \tag{3.4}$$

As is well known, the additional non-inertial term in (3.4) "vanishes" because of the anti-symmetry of W. In a Σ-system, we may drop the asterisks and assume $\mathbf{W}_{ij} \equiv 0$.

This property of invariance of Boltzmann's equation led Wang /4/ to the conclusion that material frame-indifference holds in kinetic theory. However, this principle does not follow from the invariance of the equation of momentum in the form (3.3) (nor, generally speaking, from Newton's second law on introducing the corresponding acceleration field /9/). Still less is the principle satisfied in kinetic theory. Before proving the above, we shall pay attention to the following. If $F_i \equiv 0$, then $\xi_i^{\cdot} \equiv 0$, and in passing from a Σ- to a Σ^*-system (briefly, for $\Sigma \to \Sigma^*$) additional terms appear in Boltzmann's equation since $\xi_i^{*\cdot} \neq 0$. However, for $\Sigma^* \to \Sigma^{\circ}$ this term is invariant. The equation of momentum has the same properties as well.

Precisely these properties are characteristic for the central moments of the distribution function (1.9), in particular $\mathbf{P}$ and $\mathbf{q}$ even with $\mathbf{F} \neq 0$.

4. Since the central moments of the distribution function are integrals with weights which are the products of the intrinsic velocity of the molecules $\mathbf{c} = \boldsymbol{\xi} - \mathbf{u}$, we pass in (3.1) from the variables $\boldsymbol{\xi}, \mathbf{x}, t$ to $\mathbf{c}, \mathbf{x}$ and t. By what was said above regarding its properties, the collision integral is invariant under such a transformation (see /8/). On performing certain operations /8/, and using the equation of momentum (3.3) in a Σ-system, we obtain the following Boltzmann equation for $f = f(\mathbf{c}, \mathbf{x}, t)$:

$$\frac{Df}{Dt} + c_i \frac{\partial f}{\partial x_i} + \frac{1}{\rho}\frac{\partial T_{ij}}{\partial x_j}\frac{\partial f}{\partial c_i} - V_{ij}c_j\frac{\partial f}{\partial c_i} = J(f) \tag{4.1}$$

We must stress that the terms with F_j have been cancelled, and it is out of the question to speak here about the invariance discussed in Sect.3.

The source of the inertial terms in (1.4) are V_{ij} by virtue of (1.7), and $Df/Dt \equiv \partial f/\partial t + u_j\partial f/\partial x_j$; the remaining terms are invariant under the transformations (1.1). By the formulae in Sect.1, we obtain

$$\frac{Df}{Dt} = \frac{D^*f^*}{Dt} + W_{ij}^*c_j^*\frac{\partial f^*}{\partial c_i^*}$$

Therefore Eq.(4.1) in a Σ^*-system takes the form

$$\frac{D^*f^*}{Dt} + c_i^*\frac{\partial f^*}{\partial x_i^*} + \frac{1}{\rho^*}\frac{\partial T_{ij}^*}{\partial x_j^*}\frac{\partial f^*}{\partial c_i^*} - (V^* - 2W^*)_{ij}c_j^*\frac{\partial f^*}{\partial c_i^*} = J^*(f^*) \tag{4.2}$$

The appearance of the term $2W_{ij}^*c_j^*\partial f^*/\partial c_i^*$ is physically due to the microscopic Coriolis forces which are not eleiminated by using a macroscopic equation of momentum.

Thus, as $\Sigma \to \Sigma^*$ additional inertial terms appear in Boltzmann's equation, and as $\Sigma^* \to \Sigma^\circ$ this equation is invariant: in the Σ°-system, we obtain Eq.(4.2) by replacing the asterisks by degrees.

Let us multiply Eq.(4.1) by mc_kc_m and integrate it with respect to $\mathbf{c}$. For Maxwellian molecules, when the viscosity μ is proportional to T the integration of a collision operator is carried out in explicit form, and as a result we obtain a system of equations for P_{km} (the system is open because derivatives of third-order moments occur in the equations). Of course, it is important that there are no terms with outside forces. However, the equations for P_{km}^* similarly obtained from (4.2) will contain inertial terms. Strange though it may seem, this property of the equations for the stresses led Spezialle /6/ to a "final" conclusion regarding the incorrectness of the Chapman-Enskog and Maxwell methods.

Clearly, the same results are obtained in integrating Eqs.(3.1) and (3.2) with weights: the integral of the third term on the left of (3.1) is zero unlike the corresponding integral in (3.2). It is this "asymmetry"in integrating Boltzmann's equation that explains the above properties of the central moments of the distribution function.

The established properties are found in the results of asymptotic methods ($\mathrm{Kn} \to 0$) for solving Boltzmann's equation as well. Let us consider the Chapman-Enskog method, which is most criticized in /6/. The aim of this method is to obtain an expansion for the solution of Boltzmann's equation, in the form of series

$$f \sim \sum_{n=0}^{\infty} f^{(n)}, \quad f^{(n)} \sim \mathrm{Kn}^n,$$

where $f^{(0)}$ is the local Maxwellian function. The quantities $f^{(n)}$ are functions of the intrinsic velocities $\mathbf{c}$; this is a general property of the asymptotic expansions of this equation that are external (with respect to Knudsen's layers). The series

$$P_{ij} \sim \sum_{n=0}^{\infty} P_{ij}^{(n)}, \quad q_i \sim \sum_{n=0}^{\infty} q_i^{(n)}, \quad P_{ij}^{(0)} = 0, \quad q_i^{(0)} = 0$$

which close the equation of conservation are computed from the known $f^{(n)}$. The essence of the Chapman-Enskog method consists precisely in obtaining such series; the question of how many terms should be considered is solved separately for each type of flow depending on the accuracy required. On considering $P_{ij}^{(1)}, q_i^{(1)}$, we obtain the Navier-Stokes and Fourier equations, and $P_{ij}^{(2)}$, $q_i^{(2)}$ yield the Barnett equation.

Using Eq.(4.1), the general algorithm of the Chapman-Enskog method can be written in the form

$$\sum_{m=0}^{n} \frac{D_m f^{(n-m)}}{Dt} - c_i\frac{\partial f^{(n)}}{\partial x_i} + \frac{1}{\rho}\sum_{m=0}^{n}\frac{\partial T_{ij}^{(m)}}{\partial x_j}\frac{\partial f^{(n-m)}}{\partial c_i} - V_{ij}c_j\frac{\partial f^{(n)}}{\partial c_i} - (\delta J)^{(n)} = I(f^{(n-1)}) \tag{4.3}$$

This expression is an equation for $f^{(n-1)}$; $(\delta J)^{(n)}$ is understood as the corresponding result of expanding the collision integral in series in Kn. the quantity $I(f)$ being a collision integral linearized with respect to $f^{(0)}$. The appearance of the operators $D_m Dt$ is a consequence of the exclusion, employed in the method, of the time derivatives with respect to macroquantities, using the equations of conservation (3.3) and (3.4) in a Σ-system. The action of these operators on the macroquantities is given by the formulae

$$\frac{D_0\rho}{Dt} = -\rho\nabla\mathbf{u}, \quad \frac{D_0\mathbf{u}}{Dt} = \frac{1}{\rho}(\mathbf{F} - \nabla p), \quad \frac{D_0T}{Dt} = -\frac{2T}{3}\nabla\mathbf{u} \tag{4.4}$$

$$\frac{D_{m\geqslant 1}\rho}{Dt}=0,\quad \frac{D_{m\geqslant 1}u_i}{Dt}=-\frac{1}{\rho}\frac{\partial P_{ij}^{(m)}}{\partial x_j},\quad \frac{D_{m\geqslant 1}T}{Dt}=-\frac{2}{3nk}\left(P_{ij}^{(m)}V_{ij}+\frac{\partial q_i^{(m)}}{\partial x_i}\right)$$

We note that in almost all handbooks which treat the Chapman-Enskog method, the elimination of the partial and not of the total derivatives of the macroquantities with respect to t is considered: in the stationary case this may lead to misunderstandings.

Finally, $f^{(n)}$, $n\geqslant 1$, is the sum of the quantities with the coefficient-functions c, which contain the products of tensors of different ranks, formed from the space derivatives with respect to u_i, T and ρ, the derivatives being of different orders. The tensors constructed from the derivatives with respect to frame-indifferent scalars T, ρ and $\nabla\mathbf{u}$ are frame-indifferent as well. The components of the velcoity u_i do not appear in explicit form, but only their derivatives, i.e. the components of the non-frame-indifferent tensor $\mathbf{V}$.

Not only the terms of Eq.(4.3) which contain V_{ij}, but also the operators (4.4) (and not only $D_m u_i/Dt$ as stated in /6/) are the source of inertial terms. In fact, for example by (4.4) we have

$$\frac{D_0}{Dt}\frac{\partial T}{\partial x_i}=\frac{\partial}{\partial x_i}\left(-\frac{2T}{3}\nabla\mathbf{u}\right)-V_{ki}\frac{\partial T}{\partial x_k},\quad \frac{D_0}{Dt}V_{ij}=\frac{\partial}{\partial x_j}\frac{1}{\rho}(F-\nabla p)_i-V_{kj}V_{ik}$$

where the first terms are frame-indifferent.

Consequently, as $\Sigma\to\Sigma^*$ in (4.3), only those terms which include the components of tensor $\mathbf{V}$, undergoing the transformation in accordance with rule (1.7) will be invariant. For the Σ^*-system in (4.3) we must add an asterisk everywhere at the top, and write $V_{ij}^*-W_{ij}^*$ instead of V_{ij}; because of this there appear, generally speaking, terms with a factor W_{ij}^*, which are frame-dependent. The resulting equation is invariant under $\Sigma^*\to\Sigma^\circ$ by virtue of Eq.(1.8), and $V_{ij}^*-W_{ij}^*$ is replaced by $V_{ij}^\circ-W_{ij}^\circ$.

Thus, the general algorithm of the Chapman-Enskog method is non-invariant under $\Sigma\to\Sigma^*$ and invariant under $\Sigma^*\to\Sigma^\circ$; that is the initial properties of Boltzmann's equation are maintained in the Euclidean transformations. In the Maxwell method, the expansion of the central moments of the distribution function in series in Kn is performed by the same procedures as in that of Chapman-Enskog. The Grad method is an extension of the latter to the case of arbitrary intermolecular forces. It can be similarly shown that the algorithms of these methods have the same properties during the Euclidean transformations. Finally, a segment of Hilbert's series for f can be obtained from the corresponding series in the Chapman-Enskog method by re-expanding u_i, T and ρ which the method contains, in series in Kn.

5. To a first approximation of the Chapman-Enskog method for P_{ij} and q_i we obtain the well-known expressions of Navier-Stokes and Fourier which satisfy the principle of material frame-indifference.

Let us examine in more detail the Barnett approximation (precisely this approximation was the subject of discussions in /1, 2, 5–7/). The expressions for the terms, additional to those of Navier-Stokes, for the stress tensor components have the following form in the initial frame of reference (see /12/):

$$\begin{aligned}P_{ij}^{(2)}&=\omega_1 V_{kk}\langle V_{ij}\rangle-\omega_2\left\{\left\langle F_{i,j}-\left(\frac{1}{\rho}p_{,i}\right)_{,j}\right\rangle-\right.\\&\left.\langle V_{ik}V_{kj}\rangle-2\langle\langle V_{ik}\rangle V_{kj}\rangle\right\}+\omega_3\frac{k}{m}\langle T_{,ij}\rangle-\\&\frac{\omega_4}{\rho T}\langle p_{,i}T_{,j}\rangle-\omega_5\frac{k}{mT}\langle T_{,i}T_{,j}\rangle+\omega_6\langle\langle V_{ik}\rangle\langle V_{kj}\rangle\rangle\\\omega_\alpha&=\frac{\mu^2}{p}K_\alpha,\quad \langle A_{ij}\rangle=\frac{1}{2}(A_{ij}+A_{ji})-\frac{1}{3}\delta_{ij}A_{kk},\quad A_{,ij}=\frac{\partial^2 A}{\partial x_i\partial x_j}\end{aligned}\tag{5.1}$$

In the general case, the coefficient K_α, $\alpha=1,2,\ldots,6$, depends on temperature T; here μ is the viscosity.

Let us perform the transformations $\Sigma\to\Sigma^*$ and $\Sigma^*\to\Sigma^\circ$, then, using Eq.(4.2), calculate directly $P_{ij}^{*(2)}$ in the Σ^*-system and compare the expressions obtained.

From among the tensors in (5.1) only $\mathbf{V}$ is non-frame indifferent (the quantity V_{kk} and the tensor $\langle\mathbf{V}\rangle$ are frame-indifferent). Let us pass to the Σ^*-system. In matrix notation we have

$$\begin{aligned}-\omega_2\{\langle VV\rangle+2\langle\langle V\rangle V\rangle\}&=-\omega_2^*\{\langle R^{*T}V^*V^*R^*\rangle+\\&\langle R^{*T}W^*W^*R^*\rangle-\langle R^{*T}V^*W^*R^*\rangle-\langle R^{*T}W^*V^*R^*\rangle+\\&2\langle R^{*T}\langle V^*\rangle V^*R^*\rangle-2\langle R^{*T}\langle V^*\rangle W^*R^*\rangle\}\end{aligned}$$

Taking into account the equalities

$$2\langle\langle V^*\rangle W^*\rangle=\langle V^*W^*\rangle-\langle W^*V^*\rangle,\quad W_{ij}^*=-W_{ji}^*$$

we finally obtain

$$P^{*(2)} = \Gamma^* + \omega_2{}^*B,\ B = 2\langle W^*V^*\rangle - \langle W^*W^*\rangle + 4\langle\langle V^*\rangle \times W^*\rangle \tag{5.2}$$

with $P^{*(2)} = R^*P^{(2)}R^{*T}$.

Here Γ^* denotes the right side of Eq.(5.1) with an asterisk at each variable.

In passing from Σ^* to Σ°, from the group of terms Γ^* in $P^{\circ(2)} = QP^{*(2)}Q^T$ inertial terms analogous to $\omega_2{}^*B$ are added, and $\omega_2{}^*$, V^*, W^* is replaced by $\omega_2{}^\circ$, V°, $Q^{\cdot}Q^T$, that is

$$\omega_2{}^\circ\{2\langle Q^{\cdot}Q^TV^\circ\rangle - \langle Q^{\cdot}Q^TQ^{\cdot}Q^T\rangle + 4\langle\langle V^\circ\rangle Q^{\cdot}Q^T\rangle\} \tag{5.3}$$

Let us transform the terms in the expression for B in (5.2), using the equation $V^* = Q^TV^\circ Q - Q^TQ^{\cdot}$ and formula (1.3) for W°. Taking into account $\langle Q^{\cdot}Q^TW^\circ\rangle = \langle W^\circ Q^{\cdot}Q^T\rangle$, $(QQ^T)^{\cdot} = 0$ we obtain

$$\begin{aligned}\omega_2{}^\circ QBQ^T &= \omega_2{}^\circ Q\{2\langle(Q^TW^\circ Q - Q^TQ^{\cdot})\cdot\\ &(Q^TV^\circ Q - Q^TQ^{\cdot})\rangle - \langle(Q^TW^\circ Q - Q^TQ^{\cdot})(Q^TW^\circ Q - Q^TQ^{\cdot})\rangle +\\ &4\langle Q^T\langle V^\circ\rangle Q(Q^TW^\circ Q - Q^TQ^{\cdot})\rangle\}Q^T =\\ &\omega_2{}^\circ\{2\langle W^\circ V^\circ\rangle - \langle W^\circ W^\circ\rangle + 4\langle\langle V^\circ\rangle W^\circ\rangle -\\ &2\langle Q^{\cdot}Q^TV^\circ\rangle + \langle Q^{\cdot}Q^TQ^{\cdot}Q^T\rangle - 4\langle\langle V^\circ\rangle Q^{\cdot}Q^T\rangle\}\end{aligned} \tag{5.4}$$

Adding expressions (5.3) and (5.4) we see that the additional inertial terms are cancelled out, and $P^{\circ(2)}$ takes the form (5.2) when the asterisks are replaced by degrees.

We shall obtain relation (5.2) directly from Boltzmann's equation in the Σ^*-system (4.2). The formal difference from the inertial case is that, firstly, in excluding $D_0{}^*u_i{}^*/Dt$ on account of (3.3) we must substitute $F_i{}^*$ instead of F_i, and secondly it is necessary to take into account the additional term $2W_{ij}{}^*c_j{}^*\partial f^*/\partial c_i{}^*$. Like (5.1), we find that the contribution of the first factor is $\omega_2{}^*\langle F_i{}^*\rangle_{,j}$. Hence follows the additional term of $P^{*(2)}$

$$\omega_2{}^*(2\langle W^*V^*\rangle - \langle W^*W^*\rangle)$$

To take into account the second factor it is sufficient to recognize to which terms of $P^{*(2)}$ the last term on the left in (4.1) contributes. Analysing the derivation of $P^{(2)}$ in /8/ we can see that the consequence of this term is the appearance in formula (5.1) of the last term in curly brackets for ω_2, and the last term with factor ω_6. Replacing V_{ij} by $V_{ij}{}^* - 2W_{ij}{}^*$ in these terms and taking into account the equation $\langle W^*\rangle = 0$, we find one more additional term:

$$\omega_2{}^*4\langle\langle V^*\rangle W^*\rangle = -2\omega_2{}^*\{\langle V_{mj}^*\rangle W_{mj}^* + \langle V_{mj}^*\rangle W_{mi}^*\}$$

The above expression is written in this form in /1/ with $K_2 = 2$ for Maxwellian molecules. Summing the results we again obtain formula (5.2).

The same results are found for $q_i{}^{(2)}$. In a Σ^*-system we obtain

$$\begin{aligned}q_i{}^{*(2)} = \frac{\mu^{*2}}{\rho^*}\Big\{&\frac{\theta_1{}^*}{T^*}V_{kk}^*T_{,i}^* + \frac{2}{3}\frac{\theta_2{}^*}{T^*}(T^*V_{kk})_{,i} - \theta_4{}^*\langle V_{ij}{}^*\rangle_{,j} +\\ &\Big(\frac{\theta_3{}^*}{p^*}p_{,j}{}^* + \frac{\theta_5{}^*}{T^*}T_{,j}^*\Big)\langle V_{ij}{}^*\rangle\Big\} + 2\theta_2{}^*\frac{\mu^{*2}}{\rho^*}(V^* - W^*)_{ji}T_{,j}^*\end{aligned} \tag{5.5}$$

where θ_α are analogous to K_α. Only the last term whose transformations are particularly obvious is frame-dependent.

Thus, as in Boltzmann's equation (4.1), as $\Sigma \to \Sigma^*$ there appear additional inertial terms in the expressions for the frame indifferent tensor $\mathbf{P}^{(2)}$ and in the vector $\mathbf{q}^{(2)}$, and as $\Sigma^* \to \Sigma^\circ$ the expressions for $\mathbf{P}^{(2)}$ and $\mathbf{q}^{(2)}$ are invariant.

The flows connected with these terms transport the energy and entropy without transferring the mass. However their contribution to the generation of entropy is zero, /13/. Their appearance is motivated by the fact that the Coriolis force acting on the molecules is not equal to the Coriolis "macroforce" which affects the macrovolume of the gas. Since this force is perpendicular to the velocity and does not produce the work, it does not give rise to energy or entropy.

6. The problem of the rotation of a gas as a rigid body in the Barnett approximation was discussed in /1–3/. It was maintained in /2/ that if in an inertial frame of reference Σ an isothermal gas is at rest, then in the non-inertial Σ^*-system the Barnett stresses are non-zero. However, this assertion is false: on substituting into (3.3) and (5.2) $u^* = W^*x^*$, $T^* = \text{const}$ we have $p^* = \text{const}$, $P_{ij}^{*(2)} = 0$. Similarly, using the equation of momentum to determine the derivatives of the pressure it can be proved that the Barnett stresses in an isothermal gas which rotates as a rigid body are zero in any Euclidean frame of reference, contrary to the assertions in /2/; this is a consequence of the fact that for the motion discussed, Boltzmann's equation has an exact solution which is a locally Maxwellian function of c^2 /12/. The same can be obtained for $q_i^{(2)}$.

It was emphasized in /5/ that the source of errors in /2/ discussed is the result of ignoring the obvious situation: $\mathbf{P}^{(2)}$ and $\mathbf{q}^{(2)}$ should be computed regarding the solution of a given problem, and not arbitrarily. In this connection, an analysis was given in /5/, which was more careful than that in /1/, of the solutions of Barnett's equation for a gas rotating as a rigid body at a temperature alternating with respect to the radius-vector. Such a motion is interesting in that the existence of an azimuthal heat flux follows from the last term of formula (5.5).

7. Thus, we have established that there are no internal contradictions between the initial equations and their approximate Euclidean transformations in kinetic theory. However, as was stressed in /4, 5/ it is still necessary to prove that the exact solutions of Boltzmann's equations for a real gas flow can be frame dependent. In other words, it is necessary to prove the impossibility of a situation where the non-invariant terms of $\mathbf{P}$ and $\mathbf{q}$ vanish in the solution of the kinetic equation. Let us assume that in general this situation is impossible, and consider the question of the limits of applicability of the principle of material frame-indifference.

First, we shall clarify this principle using a well-known example, see /9/. Let $P = \Phi(V, \rho, x^{\cdot}, x, t)$ where Φ is a function of five arguments. Because $\mathbf{P}$ is frame-indifferent we have $\Phi^* = R^*\Phi R^{*T}$ on (1.1), and the principle in question still requires that $\Phi(V^*, \rho^*, x^{*\cdot}, x^*, t^*) = R^*\Phi(V, \rho, x^{\cdot}, x, t) R^{*T}$. Further analysis shows that Φ can depend only on the matrix of velocity deformation $\mathrm{D} = {}^1/_2(V + V^T)$ and does not depend on $\Omega = {}^1/_2(V - V^T)$, that is $\Phi = \Phi(\mathrm{D})$ (see /9/). In other words, in conformity with the principle of material frame-indifference in a Σ^*-system the arguments of the function Φ simply "acquire" asterisks, and no additional arguments appear.

However, the kinetic theory provides examples where the function Φ depends on Ω, and therefore on the frame of reference, i.e. the above invariance does not occur, and the principle of frame-indifference has a limited applicability in the case of the motion of a gas. In this connection Murdoch /7/ proposed to widen the applicability of the principle by introducing a quantity W^* defined in (1.3). Then the constitutive relation will depend on $\Omega - W^*$, and will be invariant under Euclidean transformations. In inertial frames of reference W^* "disappears". In this sense, the results of the Chapman-Enskog method will be invariant as well.

However, in such a generalization the domain of applicability of the principle of frame indifference is limited to the case of small Knudsen numbers $\mathrm{Kn} \ll 1$, when the asymptotic method for solving Boltzmann's equation makes it possible to close the equation of conservation and thereby to pass to a macroscopic, and not to a kinetic description of the flow.

For all known types of flow, the non-invariant terms in the expressions for the stresses and heat fluxes equal Kn^2 in order of magnitude, compared with unity as $\mathrm{Kn} \to 0$. This estimate defines the domain of applicability of the principle of material frame-indifference in its usual treatment in the case of a gas.

REFERENCES

1. MÜLLER I., On the frame dependence of stress and heat flux. Arch. Rat. Mech. and Analysis, 45, 4, 1972.
2. EDELEN D.G.B. and McLENNAN J.A., Material indifference: a principle or a convenience. Intern. J. Engng Sci., 11, 8, 1973.
3. SODERHOLM L.H., The principle of material frame-indifference and material equations of gases. Intern. J. Engng. Sci., 14, 6, 1976.
4. WANG C.C., On the concept of frame-indifference in continuum mechanics and in the kinetic theory of gases. Arch. Rat. Mech. and Analysis, 58, 4, 1975.
5. TRUESDELL C., Correction of two errors in the kinetic theory of gases which have been used to cast unfounded doubt upon the principle of material frame-indifference. Meccanica, 11, 4, 1976.
6. SPEZIALLE C.G., On frame-indifference and iterative procedures in the kinetic theory of gases. Intern. J. Engng. Sci., 19, 1, 1981.
7. MURDOCH A.I., On material frame-indifference, intrinsic spin, and certain constitutive relations motivated by the kinetic theory of gases. Arch. Rat. Mech. and Analysis, 83, 2, 1983.
8. CHAPMAN S. and COWLING T., Mathematical theory of heterogeneous gases. /Russian translation/, Moscow, Izdo-vo inostr. lit., 1960.
9. TRUESDELL C., Elementary course of rational mechanics of continuous media /Russian translation/, Mir, 1975.
10. JARMAN P., Course of mechanics of continuous media /Russian translation/, Moscow, Vyssh. shkola, 1983.
11. SEDOV L.I., Mechanics of a continuous medium, Moscow, Nauka, 1983.
12. KOGAN M.N., Dynamics of a rarefied gas. Moscow, Nauka, 1967.
13. WOODS L.C., On the thermodynamics of non-linear constitutive relations in gasdynamics. J. Fluid Mech., 101, pt.2, 1980.

Translated by W.C.